MATH4030 Differential Geometry, 2016-17

Midterm

Honesty in Academic Work: *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*

Name (Chinese):______________________________

Name (English):______________________________

Student ID :__________________________________

Instruction: Answer ALL questions and show your work with explanation.

Time: 1.5 hour

Full Mark: 40

Notation: Throughout the test, $I \subset \mathbb{R}$ will always denote a connected open interval (could be unbounded), we will use "p.b.a.l." to stand for the phrase "*parametrized by arc length*", and $\mathbb{S}^2 = \{p \in \mathbb{R}^3 : |p| = 1\}$ denotes the unit sphere in $\mathbb{R}^3$ centered at origin.

Question	Score
1	
2	
3	
4	
5	
6	
Total	

Q1. *(8 points)* Let $\alpha : (-\infty, \log 2) \to \mathbb{R}^2$ be the plane curve given by

$$\alpha(t) = (e^t \sin t, e^t \cos t).$$

(i) Find a reparametrization $\beta : I \to \mathbb{R}^2$ of the curve α so that β is p.b.a.l.

(ii) Use (i) or otherwise, compute the curvature $k(t)$ of the curve $\alpha(t)$.

Q2. *(5 points)* Let $\alpha : I \to \mathbb{R}^2$ be a plane curve p.b.a.l. and suppose $p \in \mathbb{R}^2$ is a point not lying on the image of α (i.e. $p \notin \alpha(I)$). If there exists $s_0 \in I$ such that $|\alpha(s) - p| \leq |\alpha(s_0) - p|$ for all $s \in I$, show that

$$|k(s_0)| \geq \frac{1}{|\alpha(s_0) - p|}.$$

Q3. *(5 points)* Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with curvature $k(s) > 0$ for all $s \in I$. Suppose that the trace of α is contained in the unit sphere $\mathbb{S}^2$ and that α has constant torsion $\tau(s) \equiv a$. Prove that there exists some constants $b, c \in \mathbb{R}$ such that

$$k(s) = \frac{1}{b \cos as + c \sin as}.$$

(Hint: Differentiate the identity $|\alpha(s)|^2 \equiv 1$ *twice.)*

Q4. *(5 points)* Let $S \subset \mathbb{R}^3$ be a surface and $p_0 \in \mathbb{R}^3$ be a point not lying on S. Consider the smooth map $f : S \to \mathbb{S}^2$ defined by

$$f(p) = \frac{p - p_0}{|p - p_0|}.$$

Prove that $df_p : T_pS \to T_{f(p)}\mathbb{S}^2$ is not surjective if and only if $p - p_0 \in T_pS$.

Q5. *(10 points)* Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$X(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4} \right).$$

(i) Show that X is a parametrization into the sphere $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 1\}$. Which point(s) on S is not covered by X?

(ii) Is S orientable? Justify your answer.

Q6. *(7 points)* Show that

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\}$$

is a surface. Moreover, prove that S is orientable by giving a smooth unit normal vector field $N : S \to \mathbb{R}^3$.

—END OF MIDTERM TEST—

MATH 4030 Differential Geometry
Midterm
Suggested solution

1. (a)

$$\begin{aligned}
\alpha(t) &= (e^t \sin t, e^t \cos t) \\
\alpha'(t) &= e^t(\sin t, \cos t) + e^t(\cos t, -\sin t) \\
|\alpha(t)| &= \sqrt{2}e^t \\
s(t) &= \int_{-\infty}^{t} |\alpha'(u)|\, du \\
&= \int_{-\infty}^{t} \sqrt{2}e^u\, du \\
&= \sqrt{2}e^t
\end{aligned}$$

Then $t = \ln(\frac{s}{\sqrt{2}})$. When $t \to -\infty$, $s \to 0$. When $t \to \log 2$, $s \to 2\sqrt{2}$. Hence, we have an arc length parametrization $\beta : (0, 2\sqrt{2}) \to \mathbb{R}^2$ where

$$\beta(s) = \left(\frac{s}{\sqrt{2}} \sin\big(\ln(\frac{s}{\sqrt{2}})\big), \frac{s}{\sqrt{2}} \cos\big(\ln(\frac{s}{\sqrt{2}})\big) \right).$$

(b)

$$\begin{aligned}
\alpha''(t) &= e^t(\sin t, \cos t) + e^t(\cos t, -\sin t) + e^t(\cos t, -\sin t) - e^t(\sin t, \cos t) \\
&= 2e^t(\cos t, -\sin t) \\
\det(\alpha'(t), \alpha''(t)) &= \det(e^t(\sin t, \cos t) + e^t(\cos t, -\sin t), 2e^t(\cos t, -\sin t)) \\
&= \det(e^t(\sin t, \cos t), 2e^t(\cos t, -\sin t) \\
&\quad + \det(e^t(\cos t, -\sin t), 2e^t(\cos t, -\sin t)) \\
&= 2e^{2t} \det((\sin t, \cos t), (\cos t, -\sin t)) + 0 \\
&= -2e^{2t} \\
k(t) &= \frac{\det(\alpha'(t), \alpha''(t))}{|\alpha'(t)|^3} \\
&= \frac{-2e^{2t}}{2\sqrt{2}e^{3t}} \\
&= -\frac{1}{\sqrt{2}}e^{-t}
\end{aligned}$$

2. Since $p \notin \alpha(I)$,$|\alpha(s) - p|$ is differentiable.

$$\begin{aligned}
\frac{d}{ds}|\alpha(s) - p| &= \frac{\langle \alpha'(s), \alpha(s) - p \rangle}{|\alpha(s) - p|} \\
\frac{d^2}{ds^2}|\alpha(s) - p| &= \frac{\langle \alpha''(s), \alpha(s) - p \rangle + \langle \alpha'(s), \alpha'(s) \rangle}{|\alpha(s) - p|} - \frac{\langle \alpha'(s), \alpha(s) - p \rangle}{|\alpha(s) - p|^2} \frac{d}{ds}|\alpha(s) - p|
\end{aligned}$$

Since $|\alpha(s) - p|$ is maximum at $s = s_0$,

$$\begin{aligned}
\frac{\langle \alpha'(s_0), \alpha(s_0) - p\rangle}{|\alpha(s_0) - p|} &= \frac{d|\alpha(s) - p|}{ds}\bigg|_{s=s_0} = 0 \\
\frac{\langle \alpha''(s_0), \alpha(s_0) - p\rangle + 1}{|\alpha(s) - p|} &= \frac{d^2|\alpha(s) - p|}{ds^2}\bigg|_{s=s_0} \leq 0
\end{aligned}$$

Since $\frac{\langle \alpha'(s_0), \alpha(s_0) - p\rangle}{|\alpha(s_0) - p|} = 0$, $\frac{\alpha(s_0) - p}{|\alpha(s_0) - p|} = \pm N(s_0)$ where $N(p)$ is the unit normal to S at p.

$$\begin{aligned}
\langle \alpha''(s_0), \frac{\alpha(s_0) - p}{|\alpha(s_0) - p|}\rangle + \frac{1}{|\alpha(s_0) - p|} &\leq 0 \\
k(s_0)\langle N(s_0), \pm N(s_0)\rangle + \frac{1}{|\alpha(s_0) - p|} &\leq 0 \\
\pm k(s_0) &\geq \frac{1}{|\alpha(s_0) - p|} \\
|k(s_0)| &\geq \frac{1}{|\alpha(s_0) - p|}
\end{aligned}$$

3.

$$\begin{aligned}
\langle \alpha(s), \alpha(s)\rangle &= 1 \ \ \forall s \in I \\
2\langle \alpha'(s), \alpha(s)\rangle &= 0 \\
\langle \alpha'(s), \alpha(s)\rangle &= 0 \\
\langle \alpha'(s), \alpha'(s)\rangle + \langle \alpha''(s), \alpha(s)\rangle &= 0 \\
\langle \alpha''(s), \alpha(s)\rangle &= -1 \\
\langle \alpha(s), N(s)\rangle &= -\frac{1}{k(s)} \\
\frac{d}{ds}\bigg(-\frac{1}{k(s)}\bigg) &= \langle \alpha'(s), N(s)\rangle + \langle \alpha(s), -k(s)T(s) - \tau(s)B(s)\rangle \\
&= 0 - k(s)\langle \alpha(s), T(s)\rangle - \tau(s)\langle \alpha(s), B(s)\rangle \\
&= 0 - 0 - \tau(s)\langle \alpha(s), B(s)\rangle \\
&= -a\langle \alpha(s), B(s)\rangle \\
\frac{d^2}{ds^2}\bigg(-\frac{1}{k(s)}\bigg) &= -a(\langle \alpha'(s), B(s)\rangle + \langle \alpha(s), \tau(s)N(s)\rangle) \\
&= -a(0 - \frac{a}{k(s)}) \\
\frac{d^2}{ds^2}\bigg(\frac{1}{k(s)}\bigg) &= -a^2\bigg(\frac{1}{k(s)}\bigg)
\end{aligned}$$

Hence $\bigg(\frac{1}{k(s)}\bigg) = b\cos as + c\sin as$ for some constant $b, c \in \mathbb{R}$
Then $k(s) = \frac{1}{b\cos as + c\sin as}$

4. Let $v \in T_pS$, then $\exists \alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p, \alpha'(0) = v$

$$\begin{aligned} df_p(v) &= \frac{d}{dt} \frac{\alpha(t) - p_0}{|\alpha(t) - p_0|}\bigg|_{t=0} \\ &= \frac{\alpha'(t)}{|\alpha(t) - p_0|} - \frac{\alpha(t) - p_0}{|\alpha(t) - p_0|^2} \frac{\langle \alpha'(t), \alpha(t) - p_0 \rangle}{|\alpha(t) - p_0|}\bigg|_{t=0} \\ &= \frac{v}{|p - p_0|} - \frac{\langle v, p - p_0 \rangle (p - p_0)}{|p - p_0|^3} \end{aligned}$$

df_p is not surjective if and only if there exists $v \neq 0$ such that $df_p(v) = 0$ (i.e. $\frac{v}{|p-p_0|} = \frac{\langle v, p-p_0 \rangle (p-p_0)}{|p-p_0|^3}$)
So df_p is not surjective if and only if $p - p_0 \in T_pS$

5. (a)

$$\begin{aligned} \frac{dX}{du} &= \left(\frac{-4u^2 + 4v^2 + 16}{(u^2 + v^2 + 4)^2}, -\frac{8uv}{(u^2 + v^2 + 4)^2}, \frac{16u}{(u^2 + v^2 + 4)^2} \right) \\ \frac{dX}{dv} &= \left(-\frac{8uv}{(u^2 + v^2 + 4)^2}, \frac{-4v^2 + 4u^2 + 16}{(u^2 + v^2 + 4)^2}, \frac{16v}{(u^2 + v^2 + 4)^2} \right) \end{aligned}$$

Suppose $\frac{dX}{du} = k \frac{dX}{dv}$ for some constant $k \in \mathbb{R}$

$$\frac{16u}{(u^2 + v^2 + 4)^2} = k \frac{16v}{(u^2 + v^2 + 4)^2} \tag{1}$$

$$u = kv$$

$$\frac{-4u^2 + 4v^2 + 16}{(u^2 + v^2 + 4)^2} = -k \frac{8uv}{(u^2 + v^2 + 4)^2} \tag{2}$$

$$\begin{aligned} -4u^2 + 4v^2 + 16 &= -8kuv \\ -4k^2v^2 + 4v^2 + 16 &= -8k^2v^2 \\ 4k^2v^2 + 4v^2 + 16 &= 0 \text{ which has no solution} \end{aligned}$$

Hence $\frac{dX}{du}, \frac{dX}{dv}$ are linearly independent. So dX is one to one

$$\begin{aligned} &\left(\frac{4u}{u^2 + v^2 + 4} \right)^2 + \left(\frac{4v}{u^2 + v^2 + 4} \right)^2 + \left(\frac{2(u^2 + v^2)}{u^2 + v^2 + 4} - 1 \right)^2 \\ &= \frac{16u^2}{(u^2 + v^2 + 4)^2} + \frac{16v^2}{(u^2 + v^2 + 4)^2} + \frac{(u^2 + v^2 - 4)^2}{(u^2 + v^2 + 4)^2} \\ &= \frac{u^4 + v^4 + 16 + 2u^2v^2 + 8u^2 + 8v^2}{(u^2 + v^2 + 4)^2} \\ &= \frac{(u^2 + v^2 + 4)^2}{(u^2 + v^2 + 4)^2} \\ &= 1 \end{aligned}$$

$X(u,v) \in S \ \ \forall u,v \in \mathbb{R}$
Suppose $X(u_1,v_1) = X(u_2,v_2)$

$$\frac{4u_1}{(u_1^2+v_1^2+4)} = \frac{4u_2}{(u_2^2+v_2^2+4)} \tag{1}$$
$$\frac{4v_1}{(u_1^2+v_1^2+4)} = \frac{4v_2}{(u_2^2+v_2^2+4)} \tag{2}$$
$$\frac{2(u_1^2+v_1^2)}{u_1^2+v_1^2+4} = \frac{2(u_2^2+v_2^2)}{u_2^2+v_2^2+4} \tag{3}$$

By $(1)^2+(2)^2$,

$$\left(\frac{4u_1}{(u_1^2+v_1^2+4)}\right)^2 + \left(\frac{4v_1}{(u_1^2+v_1^2+4)}\right)^2 = \left(\frac{4u_2}{(u_2^2+v_2^2+4)}\right)^2 + \left(\frac{4v_2}{(u_2^2+v_2^2+4)}\right)^2$$
$$\frac{16(u_1^2+v_1^2)}{(u_1^2+v_1^2+4)^2} = \frac{16(u_2^2+v_2^2)}{(u_2^2+v_2^2+4)^2} \tag{4}$$

By (3) and (4),

$$\frac{1}{u_1^2+v_1^2+4} = \frac{1}{u_2^2+v_2^2+4} \tag{5}$$

Hence $u_1 = u_2$ and $v_1 = v_2$ by (1) , (2) & (5) . X is injective.
$X^{-1}(x,y,z) = (\frac{2x}{2-z}, \frac{2y}{2-z})$ except $(0,0,2)$. So X,X^{-1} are continuous.
Therefore X is homeomorphism and $(0,0,2)$ is not covered by X

(b) Let $F(x,y,z) = x^2+y^2+(z-1)^2$. Then $\nabla F = (2x, 2y, 2(z-1))$ and $S = F^{-1}(1)$.
Since $\nabla F(p) \neq 0 \ \ \forall p \in S$, define $N(p) : S \to \mathbb{R}^3$ by

$$N(p) = \frac{\nabla F}{|\nabla F|}(p)$$

is smooth unit normal vector field on S.
So S is orientable.

6. Let $F(x,y,z) = x^3+y^3+z^3$. Then $\nabla F = (3x^2, 3y^2, 3z^2)$ and $S = F^{-1}(1)$
$\nabla F = 0$ if and only if $(x,y,z) = 0$. So $\nabla F(p) \neq 0 \ \ \forall p \in S$
Therefore, S is a surface.
Define $N : S \to \mathbb{R}^3$ by

$$N(p) = \frac{\nabla F}{|\nabla F|}(p)$$

is smooth unit normal vector field on S
So S is orientable.